New Paths in Math

- innovative methods in math

for engineering students

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Inroduction:

This set of materials, which is the result of two years of project work of teachers and students from five European schools, has been prepared as a tool to help you to increase your students' motivation to study math and science and, at the same time, guarantee them an attractive way of gaining knowledge.

Our project goal was to improve the student's skills and competences in mathematics by improving the content of lessons with introducing the technical knowledge of our students. We also wanted to familiarise teachers of mathematics with technical knowledge that our students already have.

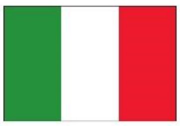
The project involved teachers and students in five partner schools:

Zespół Szkół Elektronicznych I Licealnych (Poland)

B Techniki Scholi Lemesou Gregoris Afxentiou (Cyprus)

Profesionalna Gimnazia Po Mehanoelektrotehnika I Elektronika (Bulgaria)

Escola Secundaria Campos de Melo, Covilha (Portugal)



I.I.S.S. Einaudi Casaregis Galilei (Italy)

Geogebra applets

Lessons' scenarios

Games

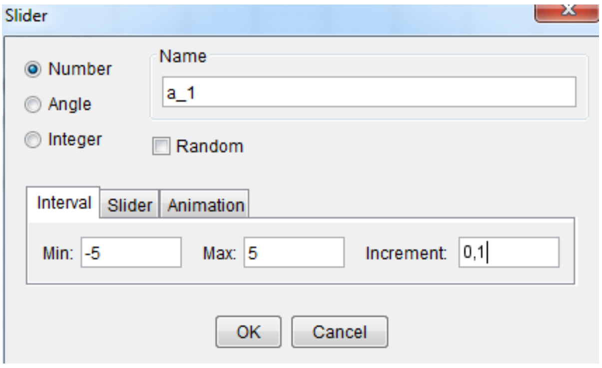
Geogebra applets

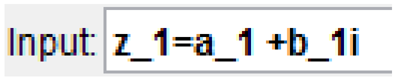
One of this project goal was to utilize ITC in teaching mathematics. On one hand we wanted to increase motivation of students with mathematics and vocational subjects and on the other hand we would like teachers to use ITC more consciously during the teaching process. Using such programs as GeoGebra for mathematics and electrical engineering will undoubtedly help to improve the teaching experience.

Complex numbers – instruction

1. Creating comlex numer's vectors

Complex numbers consist of a real part and imaginary part.
For this purpose we need 2 sliders(Fig. 1) with selected by us, the maximum and minimum value and increment(Fig. 2). Slider a_1 is a real part of comlex





number z_1 and slider b_1 is a imaginary part. If we have them, we input formula in input window z_1= a_1+ b_1i (Fig. 3)

Then we repeat everything to make a complex numer z_2, we do everything same like before with changing the name from 1 to 2.

At this time we can fix sliders and absolute their position by rightclick on them (Fig. 4), we place them so they won't disturb us, optional is we can change their colour so we can recognize them.

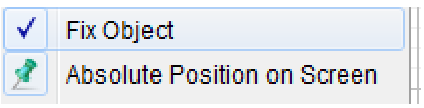
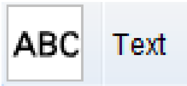
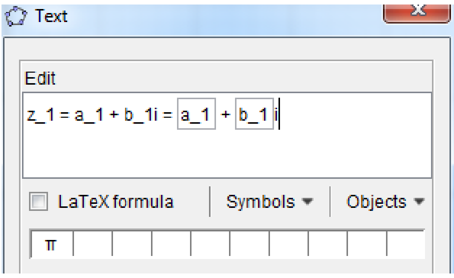


Fig. 4. Right click on slider to see this options

Next we add point A with coordinates (0,0), and we pick vector

between the points(Fig. 5). We start from point A to complex numer z_1,and the second one from A to z_2.





Under sliders a_1 i b_1 we add text field(Fig. 6). We enter here z_1= a_1 + b_1i =(in this place we choose objects a_1 and b_1 [adding i to b] from the list)(Fig. 7). We do the same thing under sliders a_2 and b_2. Colour, font and size is our choice. We fix text fields and absolute their position on the screen.

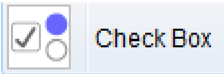
2. Creating function

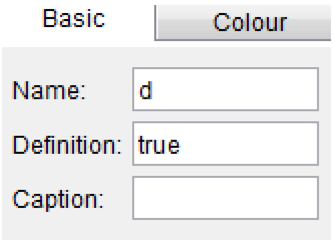
In this part we create (addition, subtraction, mutliplication and division) function.

a) Addition

For creating addition function, we add bool (Fig. 8), name him „d"(name not caption)(Fig. 9). We hide his label and add text field next to him with „Addition”.

Next in INPUT window we enter formula „zd = z_1+ z_2”, then we add





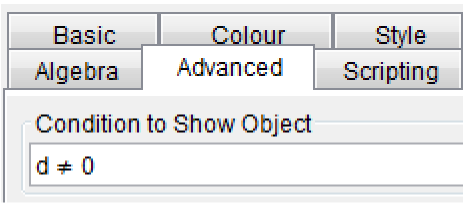
vector from A to zd. We choose point zd, and add to his caption „z_1+z_2”, label option should be set to caption.

We can add lines for better viewing how adding vectors works(Fig. 10),for this

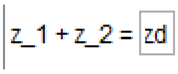
purpose we add segment from point zd to point z_1 and from zd to z_2 , we change their style for dashed line.

Now we pick addition vector, go into advanced settings and in Condition to show object (Fig. 11) we w enter „d ≠ 0”, we do the same with the dashed lines and point zd.

The last thing in addition, placing a text field under bool and name of function,



in text field we enter „ z_1 + z_2= zd” (Fig. 12), and we set d ≠ 0.

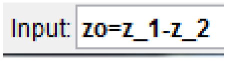


We place text and bool in right place, fix them and absolute their position.

b) Subtraction

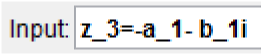
For Subtraction everything looks similar as in addition, we do nearly the same.
We add bool and name him „o”, we hide label and add text field in which we enter „Subtraction”.

We enter formula „zo = z_1– z_2”(Fig. 13), next we add vector from A to zo. In

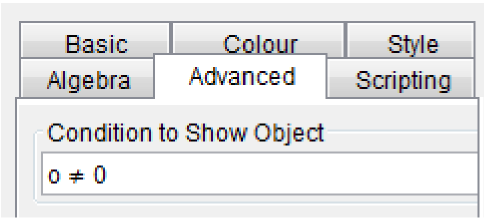


caption zo we enter „z_1– z_2”.

For the lines for help we need to create a point, in input window we enter „z_3= - a_2 –b_2i”(Fig. 14), we add vector for A to z_3, and in point caption enter „ –z_2”. We create the lines by placing segments between zo and –z_2, and between zo and z_1. We change style to dashed line.



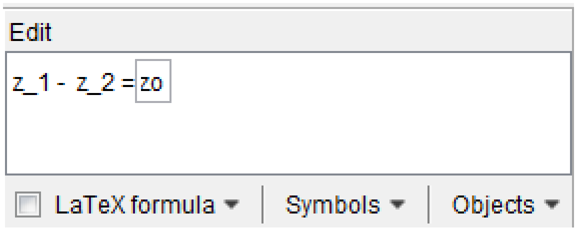
Now we pick subtraction vector and in advanced settings we set o ≠ 0(Fig. 15), same we do with points zo, -z_2 and help lines and vector –z_2.



We place text field under the bool and name of function, we enter

„ z_1-z_2=zo (Fig. 16)” and we put o ≠ 0.

We place bool and text in right place and we fix and absolute their position

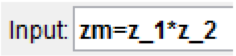


c) Multiplication

In multiplication things are easier, we create only one vector.

We place bool and name him „m”, we hide label and add text „Multiplication”.

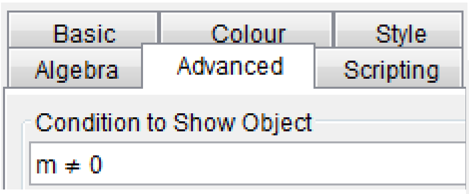
In input we enter formula „zm= z_1* z_2”(Fig. 17). We create a vector from A

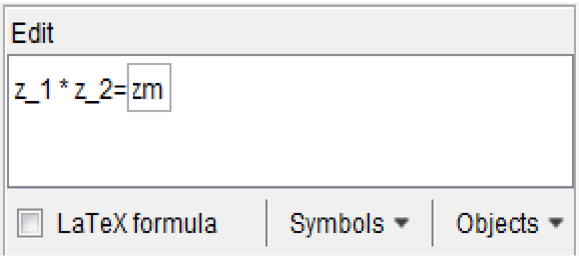


to zm. In caption of we type in „z_1*z_2”.

Next, in multiplication vector and point we go in to advanced settings and set m ≠ 0(Fig. 18).

Now under bool and name of function, we place a text field and type in „z_1*z_2 =zm”(Fig. 19) and set m ≠ 0.





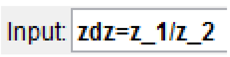
We place bool and text in right place, fix and absolute their positions.

d) Division

This function is analogous to the multiplication.

We place bool and name it „l”, we hide label and place text field and type in „Division”

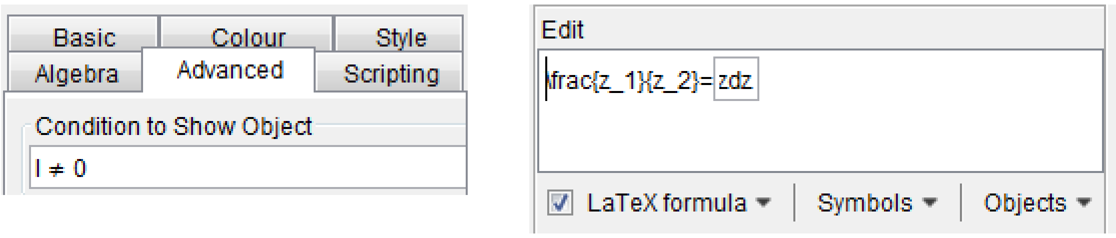
In Input we enter formula „zdz = z_1/z_2” (Fig. 20). We add vector from A to zdz. We hide zdz label,add text field, pic LaTeX formula and add fraction,



„\frac{z_1}{z_2}”. Starting point of our text we set as zdz.

In vector, created point and text we set l ≠ 0(Fig. 21).

Under bool and name of function we place text „\frac{z_1}{z_2}=zdz”

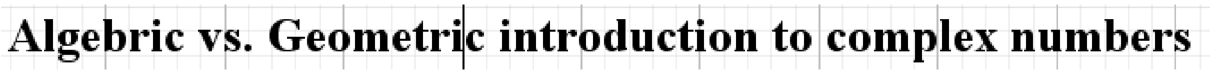


(Fig. 22) and set l ≠ 0.

We place a bool and text in right place, fix and absolute their position.

3. Visual

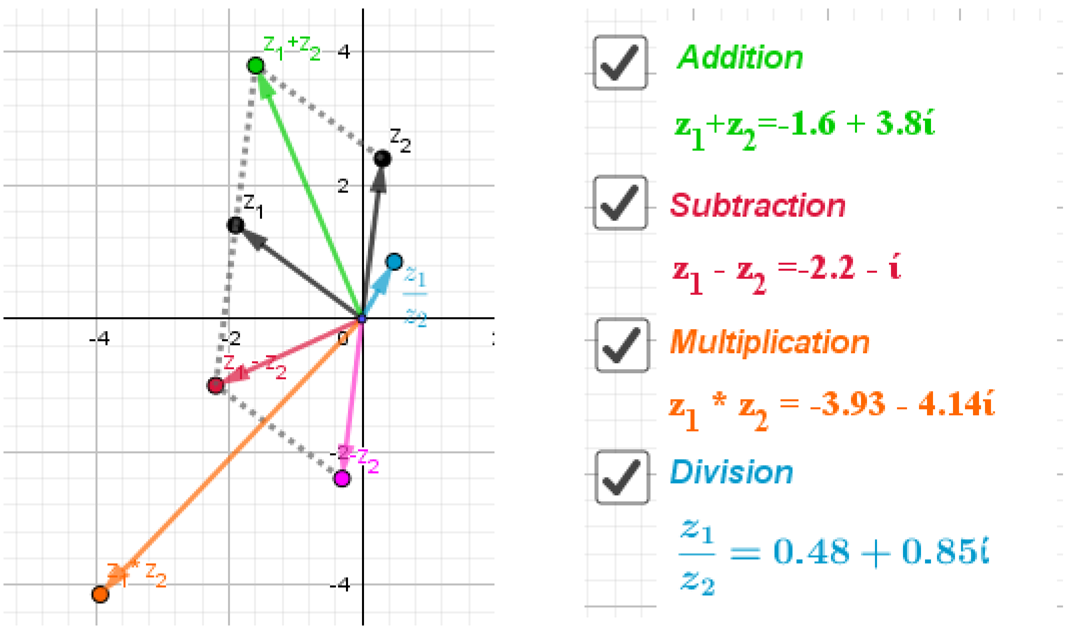
When we ended all steps, and we do not want to get out of sync during zooming in and out, we should absolute our sliders, texts and bools(we do this by right click on object and clicking absolute position).



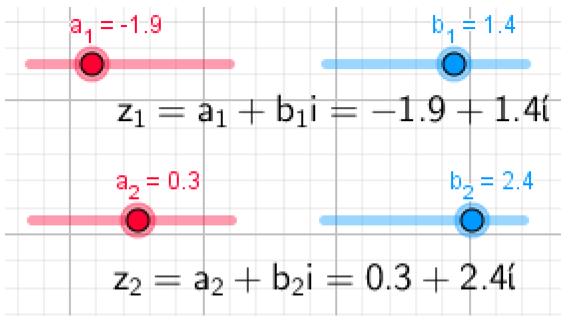
Now we can add a big title by using a text field(Fig. 23).

For a better visual effect, we can pick colours for our vectors, helpful lines, and points, the best way is picking one colour for example green for things related to addition function(Fig. 24).

Size and colour of font is a free choice(Fig. 25)(Fig. 26)



All logos, signs and pictures we place as photos but they have to had a special size.



Link to Geogebra file:

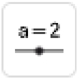
https://drive.google.com/file/d/0B4MFoJrfGsWZWFMtMDZ5MWVST1ZuYWJhV0dmMGNRWmttekJz/view?usp=sharing

Logarithm function

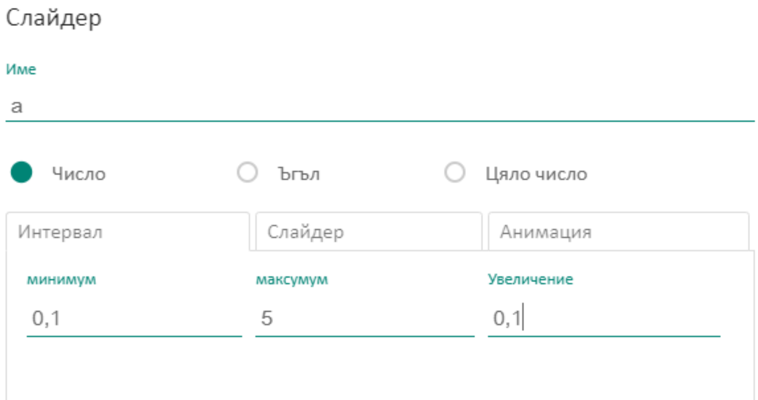
As we know the values of the logarithm, as well as the other functions, depend on its argument (**x)**, but also on the value of its basis (**a**)

$$\log_a x$$

In order for our visualization to be more universal, the values of the base will be variable, i.e. we will introduce it as a slider. Caution the base should be positive. Unfortunately, we can not set the second condition - it is unequal to1

With the button 

we set the base:



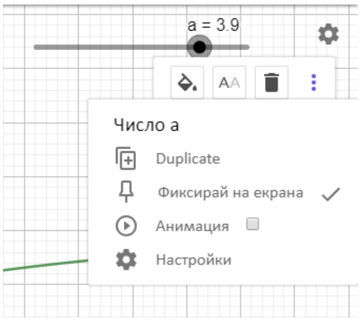
Let's draw the chart now. In algebraic mode, in the input field we enter **log(a, x)**

(To show the graph, the value of ***a*** must not be 1 so move the slider)

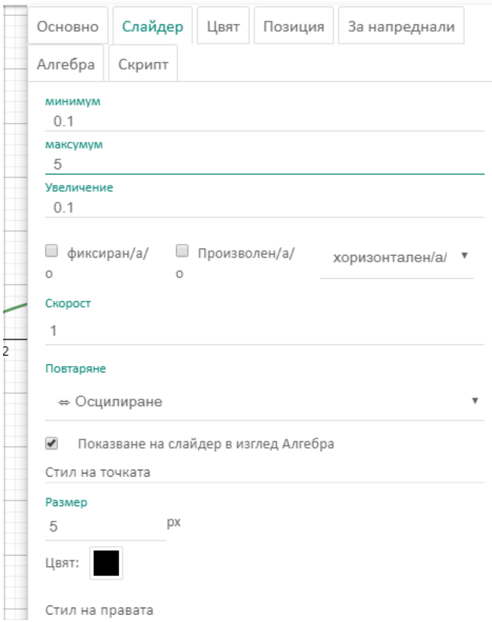
Now we'll turn on the animation button of the slider on the base and let's watch

Note that when the base value is less than 1, the graph changes symmetrically (for reciprocal basis values). Therefore, we will change the chart by setting the base values to be greater than 1, and we will enter another base $\boldsymbol{a_1}$ with values less than 1 (but larger than 0)

Change: right-click on the slider, select the three-point button from the menu, and select Settings from the new menu

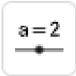


In the new window we select (if not exactly) the Slider tab

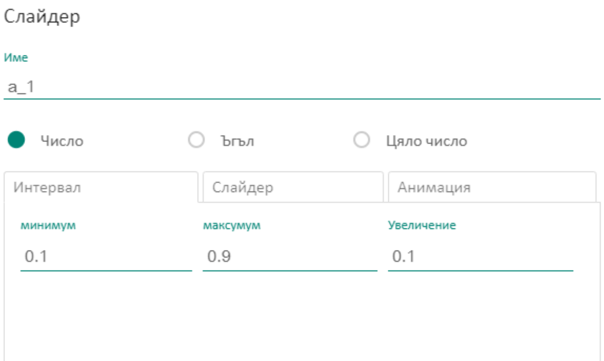


And in the Minimum box, we set a value of 1.1 and we close it

Now, as above, we will set the new base a_1

With the button 

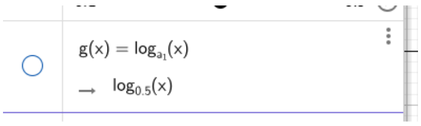
we set the base



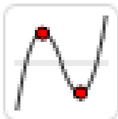
(Warning: lower index (subscript) is written using _)

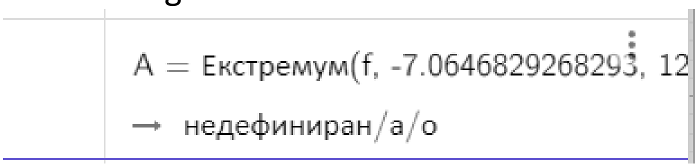
Let's draw the chart now. In algebraic mode, in the input field we enter
log(a_1, x)

To analyze in-depth, we will deactivate one of the graphs: in Algebra mode, we select the circle before the g function



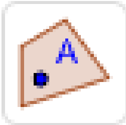
Using GeoGebra's built-in features, we will find the extremes of the function

In the Tools mode, we select the button  and point out the graph of ***f***. The conclusion we can make is that the logarithm function does not have extremes when we look at it in whole its domain. To be more credible, we switch to algebra mode and see it

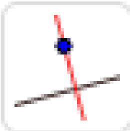


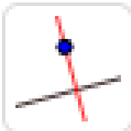
We will now analyze the change of function values when changing the argument values. This will be best illustrated by moving a point on the function graph by following its coordinates.

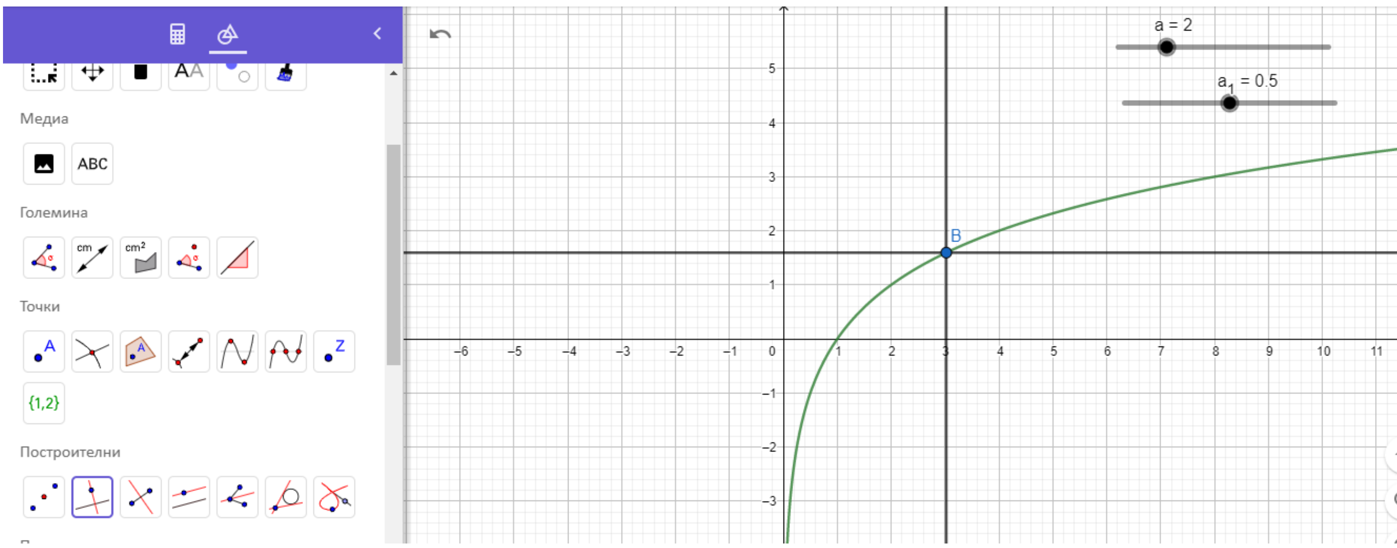
We select a point from the function graph:

In the Tools mode, we select the button  and point out the graph of function.

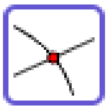
To illustrate its coordinates we will use that coordinates are determined by constructing perpendiculars to the coordinate axes:

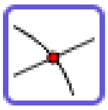
In the Tools mode, we select the button  and point to the point B and coordinate axes Ox.

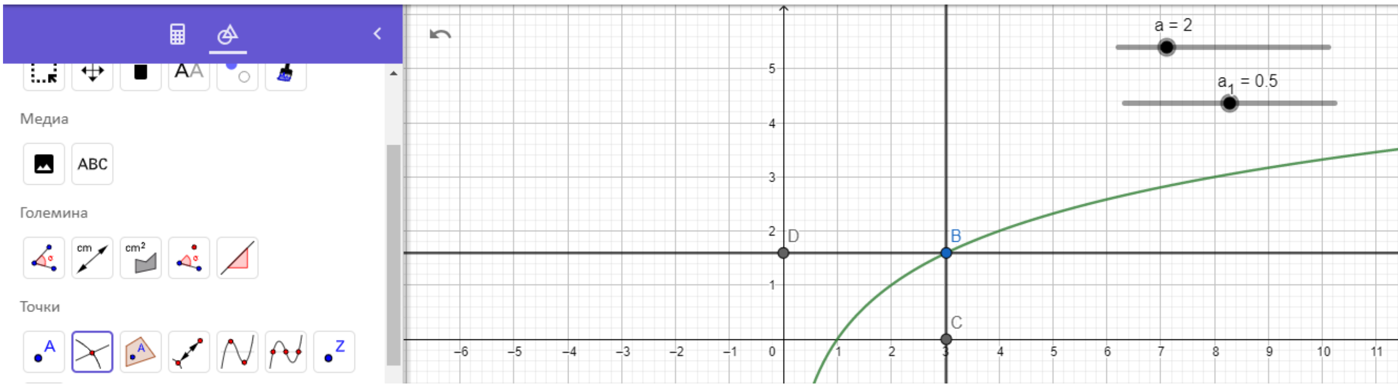
In the Tools mode, we select the button  and point to the point B and coordinate axes Oy.



The intersections of these perpendiculars with the axes are the coordinates of point B so that let us define them

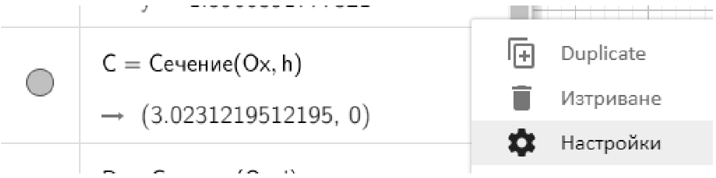
In the Tools mode, we select the button  and we point out the axis Ox and the straight line perpendicular to it.

In the Tools mode, we select the button  and we point out the axis Oy and the straight line perpendicular to it.

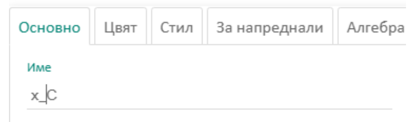


Now we will renaming them:

In Algebra mode, we point the three-point button in the order of the point (which we will rename) and select Settings

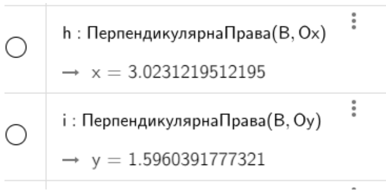


In the Basic section, enter the new name x_B (so will appear x_B)

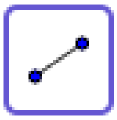


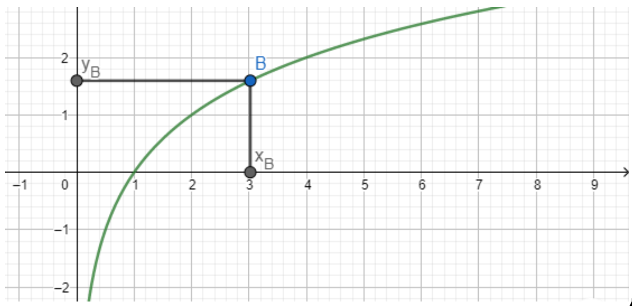
Similarly, we rename D on y_B

In order not to overload the drawing surface we will get rid of the perpendicular lines, and we will use only the segments BX_B and $B\text{У}_B$

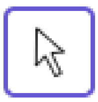
Let's turn off the straight lines 

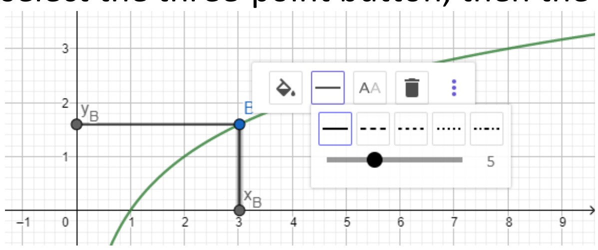
Let's draw the segments

In the Tools mode, we select the button  and point out points B and X_B, then points B and У_B



, and now we will change the line style from dense to interrupted

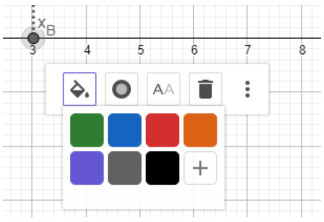
With the active arrow button  point the line on the drawing surface, select the three-point button, then the button with the line



and one of the interrupted lines

We repeat the same for $B\text{У}_B$

We recommend that you change the color of the points X_B and $У_B$ in order to monitor their movement more easily, as on this basis we will make the following conclusions



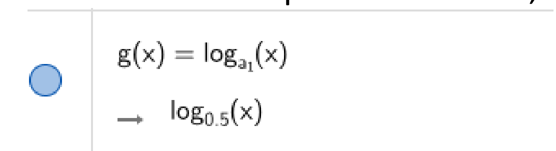
We're ready

In algebra mode for B, we select the animation button  and we follow the movement of X_B and $У_B$

<u>Conclusion</u>: With the increase of ***x*** increases ***y***, but not forget that the base here is greater than 1.

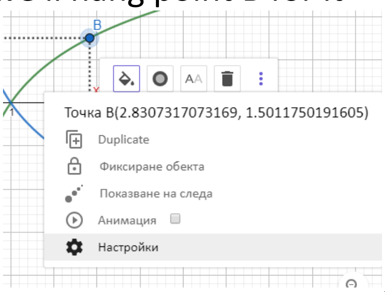
It is right to do the same test on a base less than 1, i.e. for the function ***g***

To save all of the operations above, we'll just visualize ***g***



,

we'll hang point B for it



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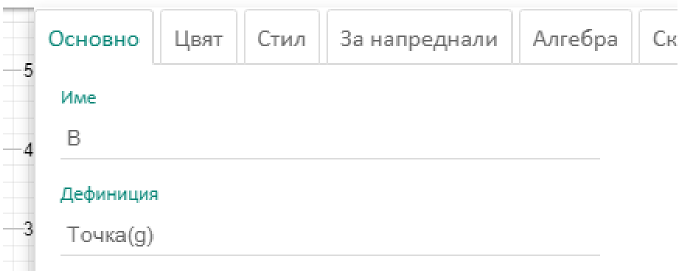
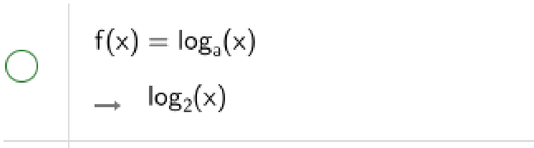
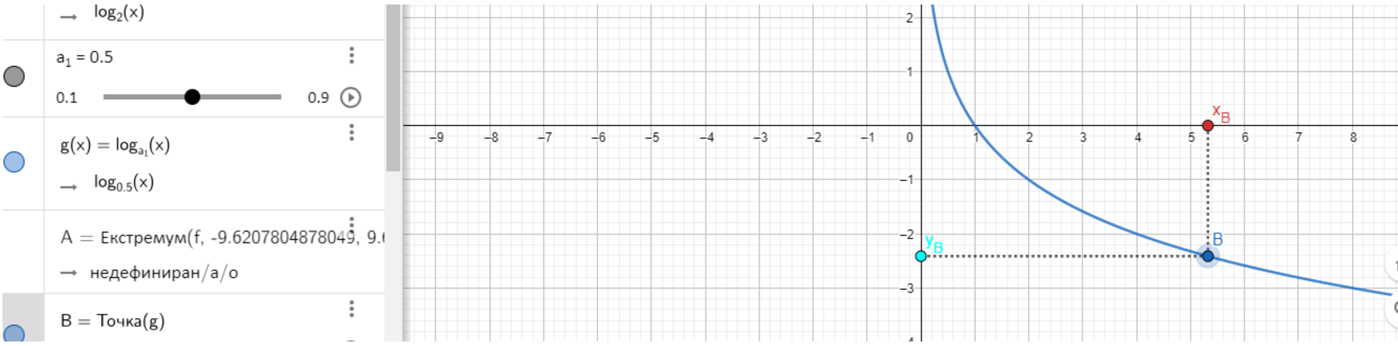


and turn off ***f*** 



Now we can follow the movement of point B again

In Algebra mode for B, we select the animation button  and we follow the movement of X_B and $У_B$

<u>Conclusion</u>: When ***x*** increases, ***y*** decreases, with here the base is less than 1.

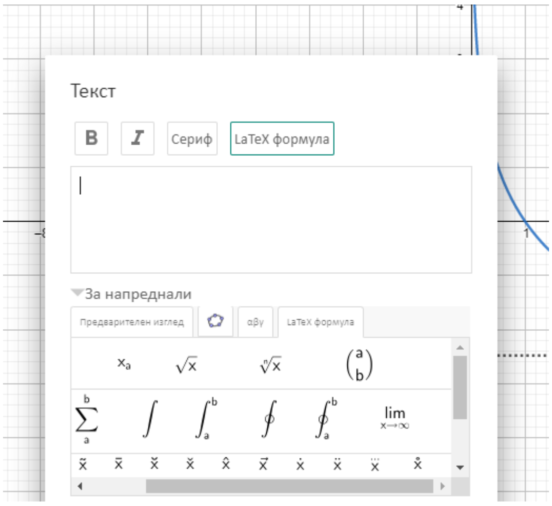
<u>Let's make a summary</u>: at base > 1, with the increase of the argument the value of the function increases, i.e. the function is increasing at base <1, with the argument increasing decreases the value of the function, i.e. the function is decreasing

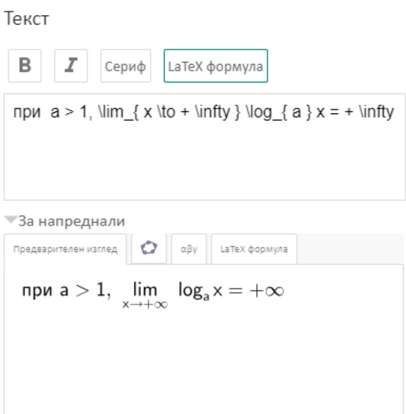
Other conclusions we can make are related to the boundary of the function

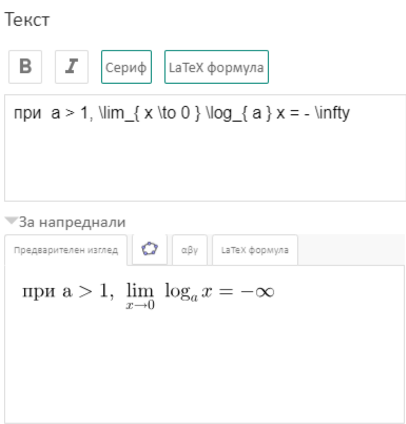
if $a > 1$, $\lim\limits_{x \to +\infty} \log_a x = +\infty$ and , $\lim\limits_{x \to 0} \log_a x = -\infty$

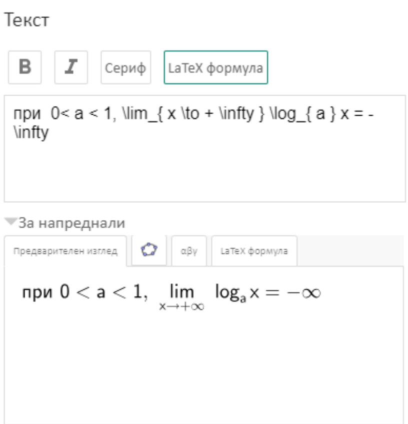
if $0 < a < 1$, $\lim\limits_{x \to +\infty} \log_a x = -\infty$ and , $\lim\limits_{x \to 0} \log_a x = +\infty$

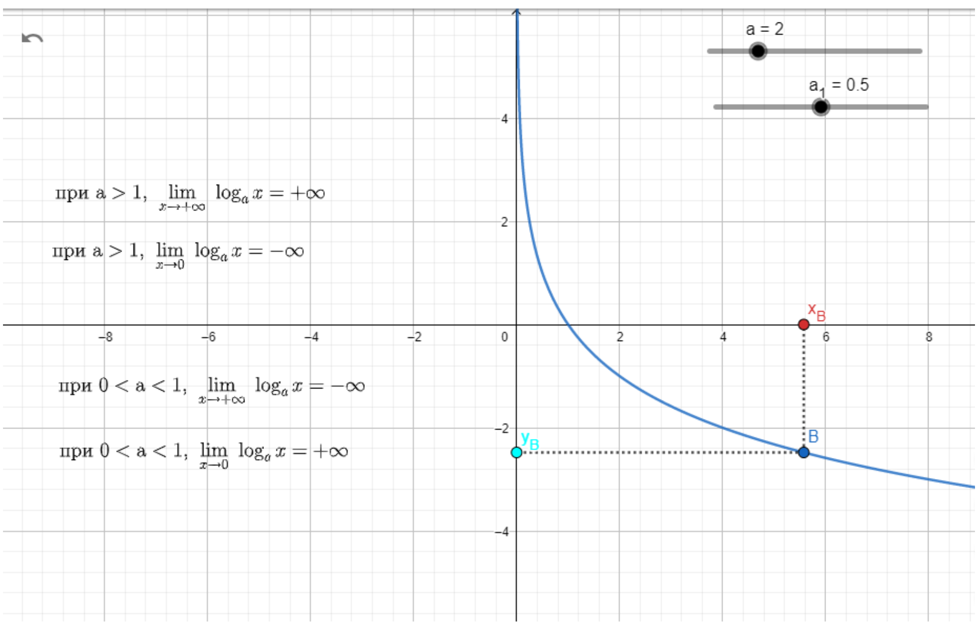
Let's get the last conclusion to write it on the drawing surface











Links to Geogebra files:

https://www.geogebra.org/graphing/dkx7apsc

https://www.geogebra.org/m/rdtg5gff

Lessons' scenarios

One of the main result of „New Paths in Math“ project are full-script lesson scenarios and teaching materials that can be used in part of the lessons. These materials are related to teaching mathematics in correlation with electronics and information technology subjects.

Lesson Plan 1

Learning Design for: Associated angles

Context

Topic: Trigonometry

Total learning time: 1 hour

Designed learning time: 50 minutes

Size of class: Around 20 students

Description: This lesson is designed to explain experimentally (using 3 Geogebra applets) the relations between the two fundamental trigonometrical functions (sine and cosine) of a given angle and the correspondent values of the same angle added to 90° and 180°.

Mode of delivery: Classroom-based

Aims

Without any theoretical basis, student will be empirically able to write down four (or even eight) fundamental trigonometrical relations

Teaching-Learning activities

Basic angles

Discuss *10 minutes* *students* *Tutor is available* *F2F*

Students must know the equivalence between the measurement of basic angles (0, 90, 180, 270 and 360°), in form of their correspondent values in radians. Knowing the classical formula of the circle C=2πr and supposing a radius of value 1 (the unit circle, so that a full circle is 2π), the teacher can guide students to fill correctly the first two columns of the given table (attached).

Linked resources

File: Basic angles.pdf

The sine and cosine functions

Practice *10 minutes* *students* *Tutor is available* *F2F*

Every student (or group of them) must open the sin-cos.ggb Geogebra file (attached).
By moving the point called "P" (better counterclockwise) and seeing how the two fundamental functions change their values (positively, negatively), the teacher can guide students to fill the last two columns of the previous table (attached again), involving basic angles.

Linked resources

File: sin-cos.ggb

https://drive.google.com/open?id=1VzL5UuujyCdQLkAOb5P9YUHX6R849Tgw

File: Basic angles.pdf

https://drive.google.com/open?id=19DrayqMQXdobk476LZBVrRSqdqxGvkH3

Adding π

Investigate *15 minutes* *students* *Tutor is available* *F2F*

Every student (or group of them) must open the add-pi.ggb Geogebra file (attached).
By moving the point "P" (better counterclockwise) on the unit circle and taking care of the color used in the constructions, students can guess the right formulas
Solutions: $\sin(a) = -\sin(a+\pi)$ or vice versa $\sin(a+\pi) = -\sin(a)$ and $\cos(a) = -\cos(a+\pi)$ or vice versa $\cos(a+\pi) = -\cos(a)$

Linked resources
File: add pi.ggb

https://drive.google.com/open?id=1I5R5BpfZO_hOSbgN_N18WREU8aEvFlC

Adding π/2

Investigate *15 minutes* *students* *Tutor is available* *F2F*

Every student (or group of them) must open the add-pi-frac-2.ggb Geogebra file (attached).

By moving the point "P" (better counterclockwise) on the unit circle and taking care of the color used in the constructions, students can guess the right formulas
Solutions: $\sin(a)=-\cos(a+\pi/2)$ or vice versa $\cos(a+\pi/2)=-\sin(a)$ and $\cos(a) = \sin(a+\pi/2)$ or vice versa $\sin(a+\pi/2) = \cos(a)$

Linked resources
File: add pi-frac-2.ggb

https://drive.google.com/open?id=13jVaIxHMBZqxXq99cDpUEnITRUHluZkt

Lesson Plan 2

Learning Design for: POLAR COORDINATES

Context

Topic: POLAR GRAPHS

Total learning time: 50

Number of students: 13

Description: THE LESSON PLAN IS DESIGNED TO DEFINE POLAR COORDINATES ON A PLANE, TO IDENTIFY THE RELATIONSHIP BETWEEN POLAR AND CARTESIAN COORDINATES AND TO PLOT POINTS AND FURTHERMORE POLAR GRAPHS

Aims

To relate the polar coordinates with Cartesian coordinates and to plot polar equations.

Tools : PowerPoint presentation
https://drive.google.com/open?id=1YmMn27GSaXfzf__FOw4rgqwk1l-6fSK9

links of Geogebra activities

PowerPoint " Polar coordinates" presentation is also attached on the project website www.newpathsinmath.eu in a section Results

Teaching-Learning activities

POLAR COORDINATES

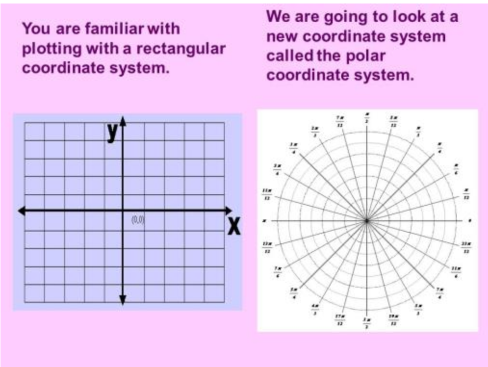
Read Watch Listen 3 minutes students Tutor is available

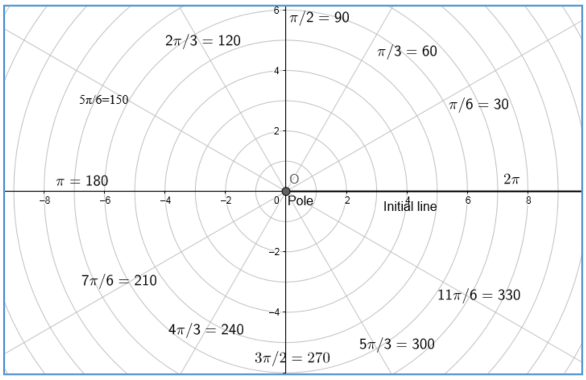
Students watch a radar screen on a PowerPoint presentation and they answer the question of what information is necessary to locate the position of a certain point. SLIDE 1

Discuss 3 minutes students Tutor is available

Following the slides of PowerPoint presentation, students mention the coordinate system that they already know. Tutor shows both polar and Cartesian coordinate systems

SLIDES 2/3

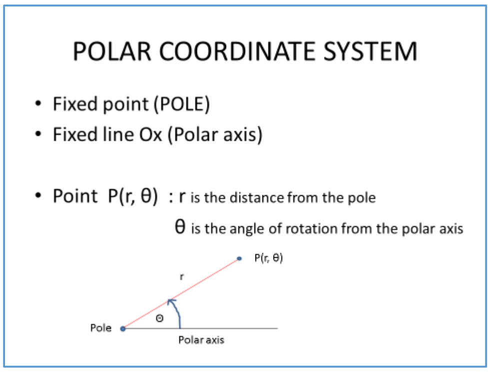


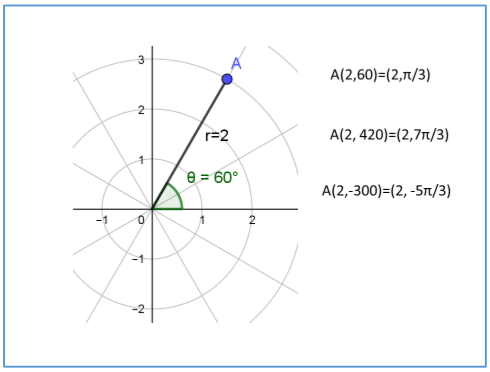


Read Watch Listen 5 minutes students Tutor is available

The teacher defines the main features (radius and angle) of polar coordinates and demonstrates how to plot points on polar plane

FIRST WITH SLIDES 4/5





AND THEN WITH GEOGEBRA ACTIVITY

https://www.geogebra.org/m/qvgac4je

Collaborate *8 minutes* *students* *Tutor is not available*

Students collaborate in plotting points in polar coordinates system(They are given polar graph paper)

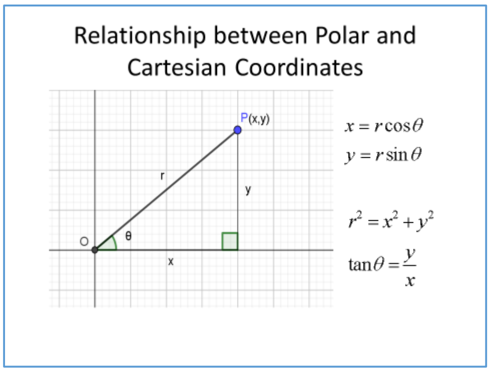
https://drive.google.com/open?id=11uP7m82Nj4ARIUzU8CzoJf2z0DEyy5Rx

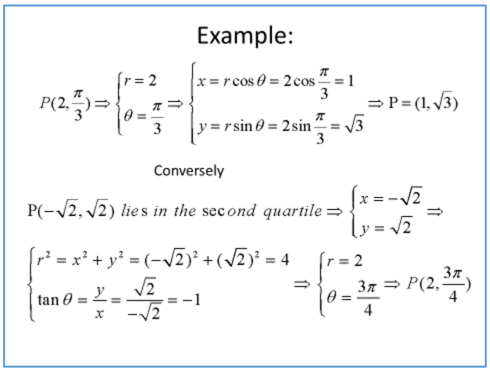
THE GEOGEBRA ACTIVITY GIVES POINTS TO BE PLOTTED
https://www.geogebra.org/m/q4z2ugbg

Investigate *5 minutes* *students* *Tutor is available*

Students investigate the relationship between polar and Cartesian coordinates. They are given guidelines by the teacher

SLIDES 6/7





Practice *10 minutes* *students* *Tutor is not available*

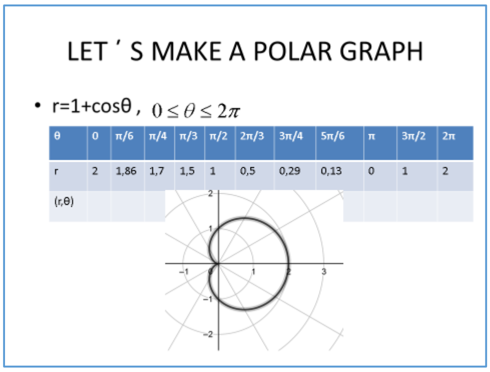
Students practice in converting polar into Cartesian coordinates and vice versa (Polar Coordinates tasks is WORD DOC and it is attached)

https://drive.google.com/open?id=1H56NAACmN7RtjhEX9F1CcNDC7T9rFMAE

Discuss *11 minutes* *students* *Tutor is available*

They discuss how they can plot the graph of a polar equation keeping in mind the corresponding method of plotting Cartesian variables of a function. They get feedback by the tutor.

SLIDE 8



GEOGEBRA ACTIVITY

https://www.geogebra.org/m/zqes7qan

Students also plot the graph on graph paper at the same time

The presentation closes with some historical information related to polar coordinates in Ancient Greece.

Lesson Plan 3

Learning Design for: Logarithm

Context

Topic: Logarithmic equations and inequalities

Total learning time: 1 hour

Designed learning time: 1 hour

Size of class: 15

Description: Students will reinforce their knowledge of the basic concepts of logarithm and the properties of logarithms; We will summarize the acquired knowledge and skills to solve different types of logarithmic equations and inequalities; We will demonstrate and teach students how to solve standard and nonstandard logarithmic equations and inequalities graphically using GeoGebra.

Aims

Enhance the basic concepts of logarithm and the properties of logarithms; Summarizing the knowledge and methods for solving the types of logarithmic equations and inequalities; Teach students to solve logarithmic equations and inequalities with varying degrees of difficulty, graphically using GeoGebra

Teaching-Learning activities

Updating knowledge

Discuss *15 minutes* *5+ students* *Tutor is available* *F2F*

Update of logarithm concepts, DM of logarithm, equation, root of equation, logarithmic equation.

Definition of logarithm

$\log_b a = c \Leftrightarrow a = b^c \quad (b>0,\ b\neq 1, a>0)$)

The teacher writes the general kind of equation of type

a) Equations of type $log_{a}f(x) = b$, a > 0, a ≠ 1.
b) A student of the class says the method of solving. The teacher reveals to the students the method he has written beforehand, and everyone checks whether there is an error, what type is, and the peculiarities of the solution of such a type of equation.

We solve with the definition of logarithm $f(x) > 0 \text{ and } f(x) = a^{b}$

Multiple examples are given for group solution

Example 1 - $\log_2 4\sqrt{2} = x$; $\log 3\sqrt{3}\ x = -2$; $\log_x 64 = 3$

The teacher writes the general kind of equation of type

b) Equations of type $log_{a}f(x) = log_{a}g(x)$, a>0, a≠1

A student of the class says the method of solving. The teacher reveals to the students the method he has written beforehand, and everyone checks whether there is an error, what type is, and the peculiarities of the solution of such a type of equation.

We solve in this way

we want one of **f (x)** or **g (x)** to be greater than **0**, so
f(x) > 0 {g(x) > 0} and f(x) = g(x)

Multiple examples are given for group solution
Example 2 - lg(x-4) + lg(x-6) = lg8

The teacher writes the general kind of equation of type

c) Equations of type $\boldsymbol{A.log_a^2 f(x) + B.log_a f(x) \; + \; C \; = \; 0 \; = \; 0}$ *,a>0,a≠1,f(x)>0*

A student of the class says the method of solving.

The teacher reveals to the students the method he has written beforehand, and everyone checks whether there is an error, what type is, and the peculiarities of the solution of such a type of equation.

We solve by entering a new unknown.

We substitute $log_a f(x) = t$ and get a square equation $At^2 + Bt + C = 0$ We find the roots (if there are) and go back to substitution to find the unknown $\boldsymbol{x}$ $\boldsymbol{log_a f(x) \; = \; t_1 \; \Leftrightarrow f(x) \; = \; a^{t_1} \;\; ; \;\; log_a f(x) \; = \; t_2 \; \Leftrightarrow f(x) \; = \; a^{t_2}}$

Multiple examples are given for group solution

Example 3 - $lg^2 x \; - \; 3.lgx \; + \; 2 \; = \; 0$

The teacher writes the general kind of equation of type d) Equations for solving which, it is necessary to apply a method of logarithm of the two parts of the equation;

Multiple examples are given for group solution

Example 4 - $x^{\frac{lgx+5}{3}} \; = \; 10^{5+lgx}$

Putting the new topic, solving tasks

Practice	*2 minutes*	*15 students*	*Tutor is available*	*F2F*

The teacher writes the general kind of equation of type e) Equations that are solved graphically by the properties of the studied functions. We will look at this method in more detail today.

Multiple examples are given for group solution

Example 5 - $log_3(4.3^x - 1) = 2.x + 1$
Students are invited to solve the task with a properly selected analytical method

Update knowledge to graphically solve equations

Discuss	*3 minutes*	*3 students*	*Tutor is available*	*F2F*

The teacher recalls that the roots of an equation f (x) = g (x) are the abscissa of of the intersection points of the graphs of f (x) and g (x).
The teacher recalls that the roots of the equation f (x) = 0 are the abscissa of the points of intersection of the graph of f (x) and the Ox axis.
The teacher recalls that the number of roots of the equation f (x) = g (x) {f (x) = 0} is equal to the number of intersections of the graphs of f (x) and g (x) {Ox axis}

Solve equations with GeoGebra

Practice	*9 minutes*	*15 students*	*Tutor is available*	*F2F*

Students stand in front of computers (one or at most two students per computer) where GeoGebra software is loaded.
The teacher hands them Worksheet 1 with the steps to complete. Students should fill in the blanks on the sheet.

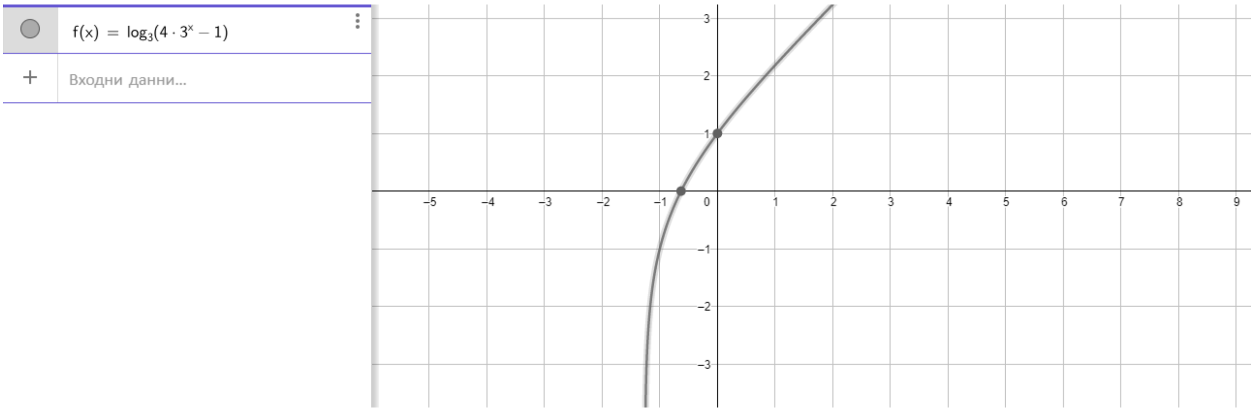
Worksheet 1

Problem is $(4.3^x - 1) = 2x + 1$

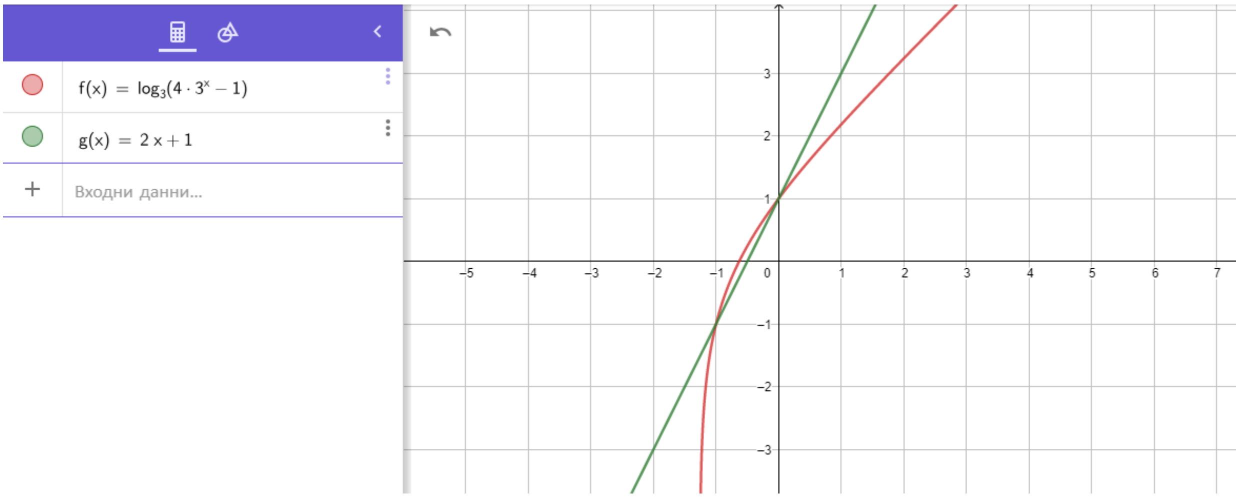
Then $f(x) = (4.3^x - 1)$, and $g(x) = 2x + 1$

We will build the graphs of f (x) and g (x), and then find their intersections.

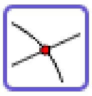
In Algebra mode we enter log(3, 4*3^x-1)

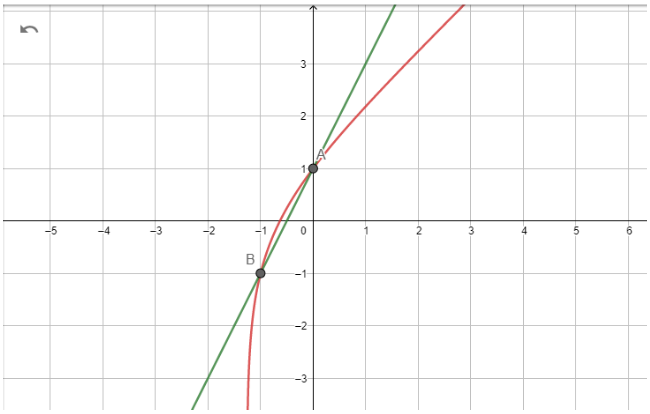


In the next field of this mode, we enter 2*x + 1



We have to find the intersections and the abscissa of their coordinates

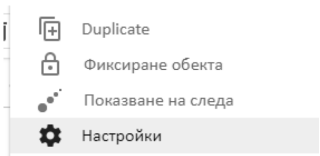
In the Tools mode, we select the button , then we point out the graph of f (x) and g (x)



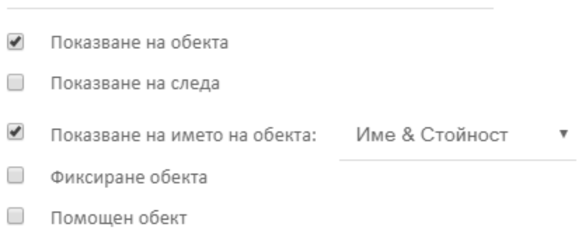
<u>Obviously the equation has</u> .... <u>roots</u>.

Now, let us also see the abscissas of p. A and p. B

With a click on A (on the drawing surface), we select the three-point button, then Settings



In the **Basic** tab, on **Show Label** we choose **Name & Value**



Similarly for the p. B

<u>Please write the answer</u>: x_1 = ..., x_2 = ...

Solve equations with GeoGebra

Practice	*7 minutes*	*15 students*	*Tutor is available*	*F2F*

The teacher presents the next challenge with equations

Example 6 At which values of the real parameter ***k*** the equation $\frac{lg\,(kx)}{lg(x+1)} = 2$ has exactly one root.

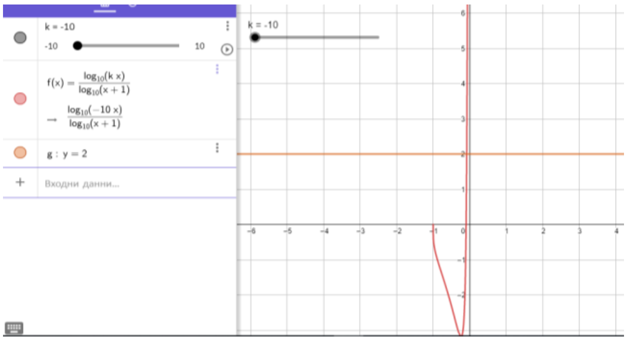
The teacher hands them Worksheet 2 with the steps to complete. Students should fill in the blanks on the sheet.

Worksheet 2

<u>Problem</u>: At which values of the real parameter ***k*** the equation $\frac{lg\,(kx)}{lg(x+1)} = 2$ has exactly one root.

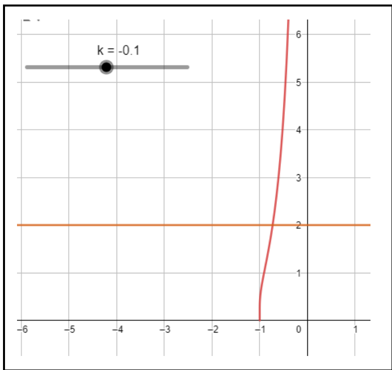
We need a slider for k with values of -10 to 10 in increments of 0.1

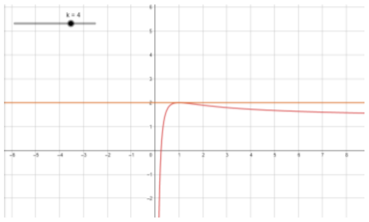
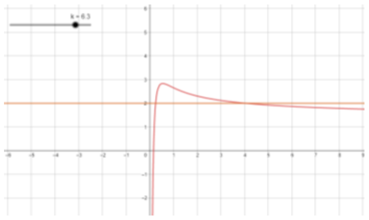
In Algebra mode we enter $f\,(x) = \frac{lg\,(kx)}{lg(x+1)}$ and $g(x) = 2$



We see that at k = -10, f and g have one intersection point, i.e. the equation f (x) = g (x) has exactly one root.

We start slowly increasing the slider

	At k = - 0.1 we have ………… root At k = 0 the graph of f (x) disappears, because is ………….

	At k> 0, the graph of f (x) appears again but does not cross the graph of g (x). We continue to increase the slider and notice that at ***k*** = 4 the two graphs again intersect at ………………. point/s
	At k values above 4, the two graphs already cross into ……….. point/s

Then **[the answer]** to our task can be as follows:

The equation has exactly one root at ……………………….

Notes

there is investigate too

Update knowledge to graphically resolve inequalities

Discuss	*3 minutes*	*5+ students*	*Tutor is available*	*F2F*

The teacher recalls that solutions of the inequality of type f (x) < g (x) are those x-spacings where the graph of the function f (x) is below the graph of the function g (x).
The teacher asks the students to determine what are the solutions to the inequality of type f (x) > g (x).
The teacher asks the students to determine what part of the graphs should we watch if the inequalities are not rigid, ie equality is allowed.

Solve inequalities with GeoGebra

Practice	*7 minutes*	*15 students*	*Tutor is available*	*F2F*

The teacher presents the next challenge with an inequality

Example 7 - $\sqrt{log_2(\frac{3-2x}{1-x})} < 1$.

The teacher hands them Worksheet 3 with the steps to complete. Students should fill in the blanks on the sheet.

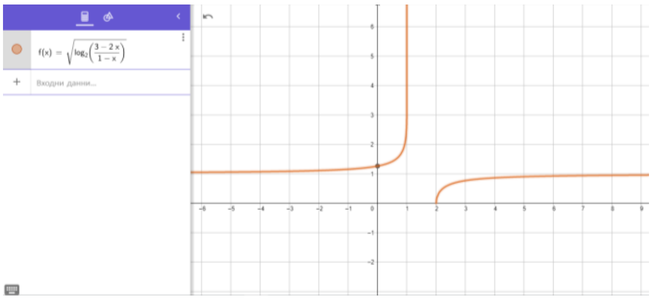
Worksheet 3

Problem : $\sqrt{log_2(\frac{3-2x}{1-x})} < 1$

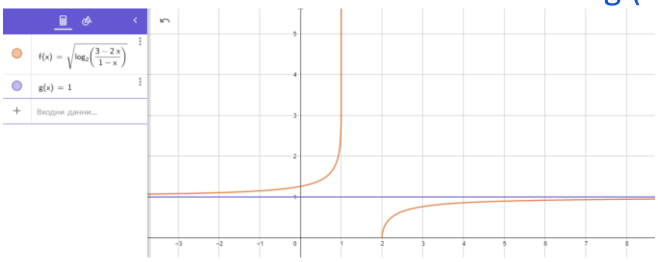
Then $f(x) = \sqrt{log_2(\frac{3-2x}{1-x})}$, and $g(x) = 1$

We will build the graphs of f (x) and g (x), and then find their intersections.

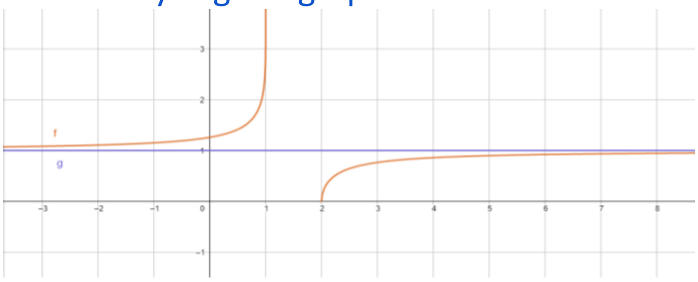
In Algebra mode we enter sqrt(log(2, (3-2x)/(1-x)))



In the next row of the section we enter g (x) = 1



After analyzing the graphs



it is obvious that the graph of f (x) is below the graph of g (x) for values of x

…………………………, i.e. **<u>the answer is</u>** x ∈ …………………….

Notes: there is investigate too

Self-writing and solving a problem.

Produce	*2 minutes*	*15 students*	*Tutor is not available*	*F2F*

The teacher requires each student to write an equation that would be difficult to solve using the analytical method.

Produce	*9 minutes*	*15 students*	*Tutor is not available*	*F2F*

Students exchange sheets of written equations with each other and everyone tries to find the answer through GeoGebra.

Set homework and receiving feedback

Read Watch Listen	*2 minutes*	*15 students*	*Tutor is available*	*F2F*

The teacher requires each student to write an inequality that would be difficult to solve using the analytical method and to find the answer through GeoGebra.

Produce	*1 minute*	*15 students*	*Tutor is available*	*F2F*

The teacher invites: students who are scared of the lesson to kneel to their chairs; students for whom the lesson was boring to put their heads on the table; students who are happy with the lesson to applaud.

Lesson Plan 4

Subject: Mathematics

Topic: Logistic model

Age of students: 17-18

Time: 120 (60 + 60) minutes

General teaching aims:

- Identify mathematical models in real problems;

- Develop skills with ICT;

- Use Geogebra Software to obtain functions that translate the problem under study;

- Using Geogebra Software to achieve the desired regression;

- Share, discuss and draw conclusions about the study of the logistic model.

Content aims:

After completing the lesson students will be able to:

- Apply mathematical models to real problems;

- Use the new technologies, through Geogebra Software, to solve real problems.

Teaching Learning activities:

Logistic model (60 minutes)

5 minutes – The teacher begins by explaining that the exponential functions and logarithmic functions are excellent models for the study of physical and social phenomena. Ask students to organise themselves in 5 groups.

Research -15 minutes – students are asked to investigate the application of exponential and logistic models in real problems.

Collaboration – 30 minutes – Each group will share the examples found in the exponential and logistic models in real situations.

Discussion – 10 minutes – The teacher will provide discussion on the examples found in terms of their similarities and characteristics.

Exponential regression and logistics with Geogebra (60 minutes)

Production - 15 minutes - the teacher will ask his/her students to organize in 5 groups. Then the teacher gives the worksheet to each student and tells them that they will use Geogebra Software to study and find answers to real questions.

The teacher begins by exemplifying/ solving question 1 of the worksheet that is about the search for the best function that characterizes a study on a colony of bacteria.

The teacher uses the computer that is connected to the overhead projector to show all students the necessary steps to do the exercise.

Production - 15 minutes – By using Geogebra Software, each group must perform task 3 or task 4 of the worksheet on exponential regression and logistics.

Collaboration – 20 minutes – The tasks will be presented by the groups, indicating suggestions for resolution.

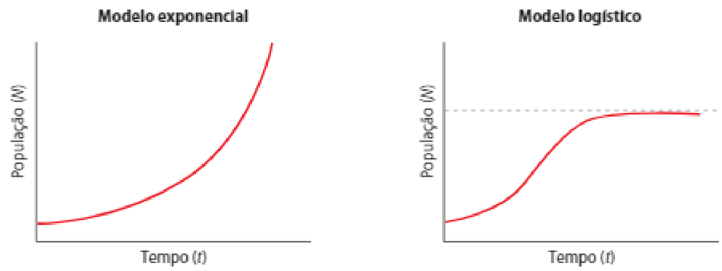
Production – 10 minutes – conclusions on exponential regression and logistics are drawn.

Logistic model

The exponential function and logarithmic function are often excellent models for physical, biological and social phenomena. One of the most usual applications of the exponential function is the study of population growth.

However, reality shows that populations do not always follow the exponential growth pattern, because this growth depends not only on the existing population but also on other factors that limit it, such as the physical space or the availability of food.

Thus, in the study of population growth, it is normallyconsidered two models: the exponential model and the logisticmodel.



In the exponential model, the population increases without limits. In the logistic model, more realistic, there is a rapid initial growth of the population, followed by a stabilization.

The function on which the logistic model is based is the logistic function.

The logistic function is given by $f(x) = \frac{c}{1+a \cdot e^{-bt}}$, in which a, b and c are positive constants and t is the variable time.

The logistic model, besides being a population model, is also widely used as a model for the sales of a given product, for the propagation of a virus, etc.

Exponential regression and logistics

Task I

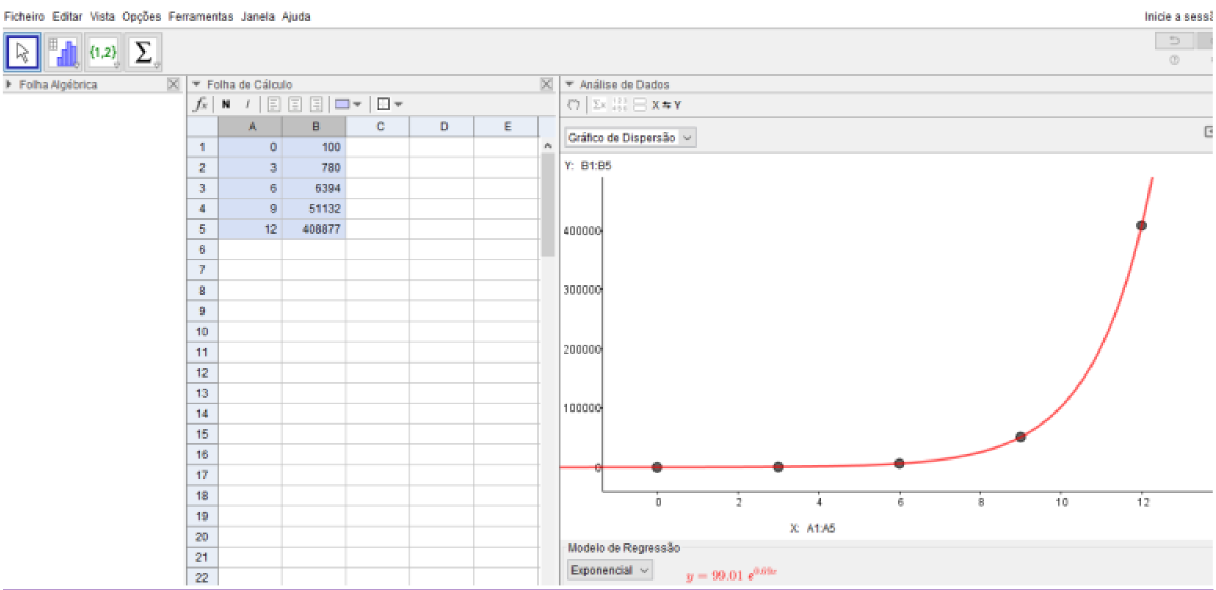
A group of researchers conducted a study on a bacterial colony in order to verify its development. The evolution of the number of bacteria was recorded in the following chart:

t	0	3	6	9	12
b=f(t)	100	780	6394	51132	408877

t is the number of hours since the creation of the culture and *b* the number of bacteria that make up the crop.

Using the Geogebra Software we can represent the datathrough a cloud of points. Taking into account the obtained cloud, the model that best fits this set of points is the exponential model of the type $f(t) = a \cdot e^{kt}$ or $f(t) = a \cdot b^{t}$

So: We can look for the best function to get the desired regression.



Task II

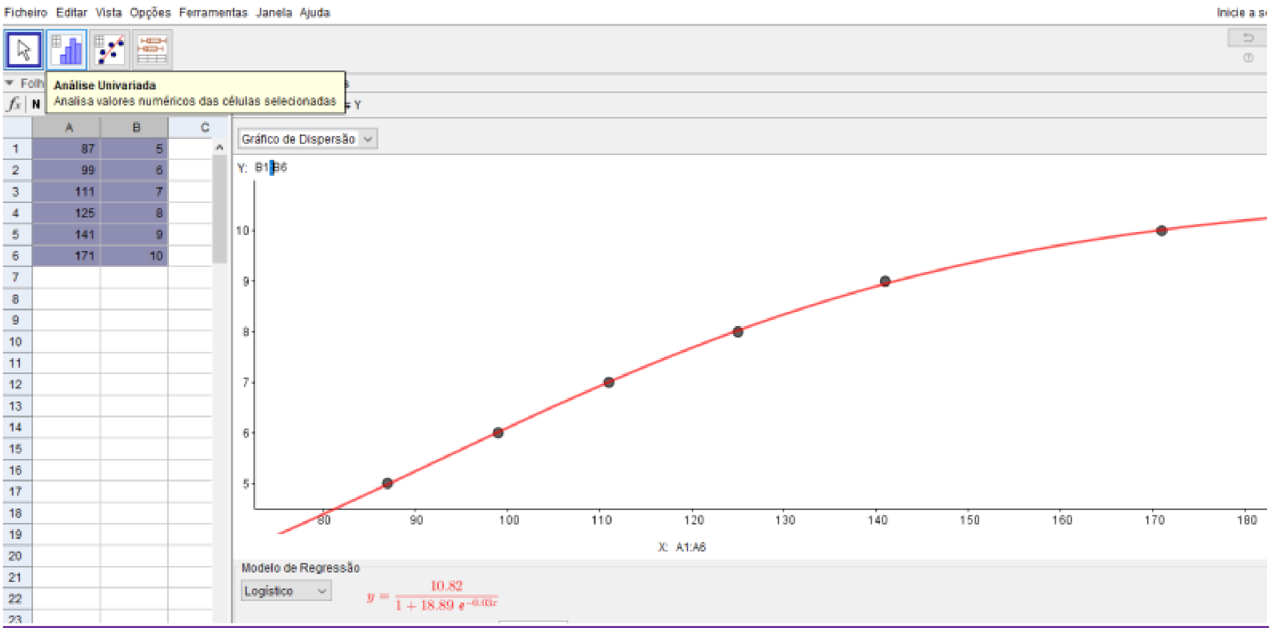
Now, consider the following situation:

The following chart was presented in a United Nations study, published in the New York Times, which argues that the growth of the world's population is slowing down.

Year	1927	1960	1974	1987	1999	2011	2025	2061	2071
t	27	60	74	87	99	111	125	141	171
p	2	3	4	5	6	7	8	9	10

t represents the number of years elapsed since 1900 and p represents the world population in billions of people.

Using the Geogebra Software, obtain the population model defined by the United Nations study.



Task III

In Portugal, as in the rest of the world, at the end of the twentieth and early 21st centuries, the cell phone became a mass product and it had a huge expansion. In the table on the right, the number of mobile phones in Portugal can be observed for every thousand inhabitants, between 1994 and2001, according to Pordata *.

1.1. Using Geogebra Software, represent this data through a scatter cloud and indicate the model that best applies to the data.

1.2. Use the model you found in the preceding exercise to indicate the number of mobile phones per thousand inhabitants expected to 2005.

1.3. Can this model be valid for many years? Justify.

Year	No. of mobile phones (in thousand inhabitants)
1994	17
1995	34
1996	66
1997	145
1998	302
1999	456
2000	645
2001	804

Task IV

2. In the table on the right, based on the data from PORDATA, you can find the number of students enrolled in Higher Education in Portugal, between 1985 and 2003.

Year	No. of students enrolled
1985	102145
1986	106216
1987	117128
1988	123507
1989	135935
1990	157869
1991	186780
1992	218317
1993	246082
1994	269348
1995	290348
1996	315415
1997	334125
1998	347473
1999	356790
2000	373745
2001	387703
2002	396601
2003	400122

2.1. Using the Geogebra software, represent this data through a scatter cloud and indicate the model that best applies to the data.

2.2. According to the model, what year do you expect to have 430 000 students enrolled?

* Database of Contemporary Portugal.

GAMES

While working on this project we noticed that game-based learning approach also increase students motivation to learn. Therefore we decided to share some of them in this publication. In our opinion these examples are the most appreciate for use and can be used by teachers without too much preparation time.

GUESS THE NUMBER

Genoa (Italy), April 2019

How to perform the trick

Invite each member of the audience to think of a number between 1 and 99. These numbers must be kept secret, for example writing them down on a blank paper, and then folding it.

Thus, progressively show the audience the seven tables reproduced here below:

1	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**
21	**23**	**25**	**27**	**29**	**31**	**33**	**35**	**37**	**39**
41	**43**	**45**	**47**	**49**	**51**	**53**	**55**	**57**	**59**
61	**63**	**65**	**67**	**69**	**71**	**73**	**75**	**77**	**79**
81	**83**	**85**	**87**	**89**	**91**	**93**	**95**	**97**	**99**

Table A

2	**3**	**6**	**7**	**10**	**11**	**14**	**15**	**18**	**19**
22	**23**	**26**	**27**	**30**	**31**	**34**	**35**	**38**	**39**
42	**43**	**46**	**47**	**50**	**51**	**54**	**55**	**58**	**59**
62	**63**	**66**	**67**	**70**	**71**	**74**	**75**	**78**	**79**
82	**83**	**86**	**87**	**90**	**91**	**94**	**95**	**98**	**99**

Table B

4	**5**	**6**	**7**	**12**	**13**	**14**	**15**	**20**	**21**
22	**23**	**28**	**29**	**30**	**31**	**36**	**37**	**38**	**39**
42	**45**	**46**	**47**	**52**	**53**	**54**	**55**	**58**	**61**
62	**63**	**68**	**69**	**70**	**71**	**76**	**77**	**78**	**79**
84	**85**	**86**	**87**	**92**	**93**	**94**	**95**		

Table C

8	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**24**	**25**
26	**27**	**28**	**29**	**30**	**31**	**40**	**41**	**42**	**43**
44	**45**	**46**	**47**	**56**	**57**	**58**	**59**	**60**	**61**
62	**63**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**
88	**89**	**90**	**91**	**92**	**93**	**94**	**95**		

Table D

16	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
26	**27**	**28**	**29**	**30**	**31**	**36**	**37**	**38**	**39**
44	**45**	**46**	**47**	**52**	**53**	**54**	**55**	**60**	**61**
62	**63**	**80**	**81**	**82**	**83**	**84**	**85**	**86**	**87**
88	**89**	**90**	**91**	**92**	**93**	**94**	**95**		

Table E

32	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**
42	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**
52	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**	**61**
62	**63**	**96**	**97**	**98**	**99**				

Table F

64	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**
74	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**
84	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**
94	**95**	**96**	**97**	**98**	**99**				

Table G

(Alternatively, you can print them on white paper and then glue them on a stiff cardboard and cut them out along the edges).

Ask the audience to indicate you the tables where their numbers are, and then - taking a quick look at the tables they indicated you - without hesitation, guess the number they thought of!

Trick explanation

Being able to find the number chosen by the audience, you simply need to add the values that appear in top left cell of every table they indicate you.

For example, if someone tells you that his number appears on the tables B (where 2 is in first cell), D (where in the first cell there is an 8), G (where 64 is in the first cell) the number he thought of is 74, given by $\mathbf{2+8+64 = 74.}$

The Mathematics behind this game

Each of the seven tables is related to a specific power of 2, in the manner explained below:

Table	Power of two
A	$2^0 = 1$
B	$2^1 = 2$
C	$2^2 = 4$
D	$2^3 = 8$
E	$2^4 = 16$
F	$2^5 = 32$
G	$2^6 = 64$

You'll find a number (between 1 and 99) printed on each of the previous tables, only if its encoding in binary system gives a "1" digit in the column of the power of two related to it.

To understand this, you can verify that:

- on table A these numbers are printed: 1, 3, 5, ... 99, that is all the numbers whose binary coding contains a "1" digit in the last column (the one associated with the power $2^0 = 1$): **1**, 1**1**, 10**1**, ..., 110001**1**
- on table B you can find: 2, 3, 6, ... 99. This means all those numbers whose binary coding contains a 1 in the penultimate column (the one associated with the power $2^1 = 2$): **1**0, **1**1, 1**1**0 ..., 11000**1**1, and so on

In this way, if you know which tables contain a certain number, you can also know which columns of its binary encoding contain the digit "1" (and which, by exclusion, the digit "0"). However, you won't need to perform the decoding operation from the binary system to guess the value of numbers thought by the audience. In fact, to speed up the consultation, the numbers printed in the previous tables are arranged in ascending order and each card has (in the upper left corner) the value of the power of 2 associated with it. As a result, binary decoding is performed automatically, adding the numbers that appear in the upper left cell of the selected cards.

For example, if the number to be guessed appears on cards B, D and G, its encoding in binary must contain the digit "1" in the positions: 2nd, 4th and 7th (counting from the right) and consequently, corresponds to: 1001010; therefore, its decimal value is equal to:

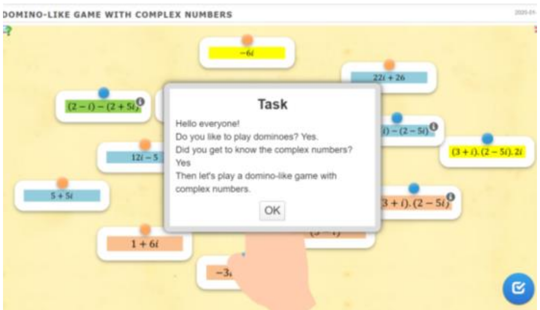
$$2^1 + 2^3 + 2^6 = 2 + 8 + 64 = 74$$

As you can see, you can obtain this value by adding the numbers that appear at the top of left in the cards concerned (B, D and G).

FUNNY GAME (just click on the picture)

or scan this

DOMINO-LIKE GAME WITH COMPLEX NUMBERS
(just click on the picture)



or scan this

If you would like to find more materials and project results please visit the project website www.newpathsinmath.eu